ISRAEL JOURNAL OF MATHEMATICS, Vol. 37, Nos. 1–2, 1980

ON THE PRIMARINESS OF THE POULSEN SIMPLEX SPACE

BY
WOLFGANG LUSKY

ABSTRACT
We show that the Poulsen simplex space is primary by investigating special monotone bases in L_1-predual spaces.

1. Introduction

Assume all Banach spaces to be real. A Banach space X is called *primary* if for every bounded projection $P:X\to X$ we have that PX or (id-P)X is isomorphic to X. Let S be the Poulsen simplex, i.e. the metrizable compact Choquet simplex whose extreme point set ex S is dense in S. It is well known that S is uniquely determined (up to affine homeomorphisms) by this property ([4]). The corresponding L_1-predual space $A(S)=\{f:S\to\mathbf{R}\mid f \text{ affine, continuous}\}$ is maximal in the sense that every separable L_1-predual space X is isometrically isomorphic to a subspace Y of $A(S)$ with contractive projection $P:A(S)\to Y$. We can even assume that (id-P)$(A(S))$ is isomorphic to $A(S)$ ([7]). Furthermore, $A(S)$ is isomorphic to the Gurarij space G which is uniquely defined by the following property: Whenever $E\subset F$ are finite dimensional Banach spaces and $T:E\to G$ is an isomorphism then there is an extension $\tilde{T}:F\to G$ of T to an isomorphism with $\|\tilde{T}\|\|\tilde{T}^{-1}\|\leqq(1+\varepsilon)\|T\|\|T^{-1}\|$ for every given $\varepsilon>0$. In contrast to these spaces $A(S)$ and G, $C(\Delta)$, Δ the Cantor set, has a minimality property: Every separable L_1-predual space X, whose dual X^* is non-separable, contains an isometric copy Z of $C(\Delta)$ with contractive projection $P:X\to Z$ ([3]). It is well known that $C(\Delta)$ and moreover all $C(K)$-spaces with metrizable compact K are primary ([1], [5]).

In the present paper we show the primariness of $A(S)$ and hence of G. Note that $A(S)$ is not isomorphic to any complemented subspace of a $C(K)$-space ([2]).

Received October 16, 1979 and in revised form February 20, 1980

THEOREM I. *Let $Q : A(S) \to A(S)$ be a bounded linear operator and let $\varepsilon > 0$. Then for $\tilde{Q} = \text{id-}Q$ or for $\tilde{Q} = Q$ there is a subspace $Y \subset A(S)$ and a contractive projection $R : A(S) \to Y$ such that*

(i) $Y \cong A(S)$,

(ii) $1/(2+\varepsilon)\| y \| \leqq \| R\tilde{Q}y \| \leqq \| \tilde{Q}y \|$ *for all* $y \in Y$,

(iii) *there is a projection* $R_1 : A(S) \to \tilde{Q}Y$ *with* $\| R_1 \| \leqq (2+\varepsilon)\| \tilde{Q} \|$.

THEOREM II. *Let $Q : A(S) \to A(S)$ be a bounded projection. Then, for $\tilde{Q} = Q$ or for $\tilde{Q} = \text{id-}Q$, $\tilde{Q}A(S)$ is isomorphic to $A(S)$, i.e. $A(S)$ is primary.*

COROLLARY. *There is a separable simplex space X, whose dual is non-separable, which is not primary. Hence X is neither isomorphic to $C(\Delta)$ nor to $A(S)$.*

PROOF. There is a simplex space Y whose dual is l_1 ([2]) which is not isomorphic to any complemented subspace of any $C(K)$-space. Define $X = (C(\Delta) \oplus Y)_{(\infty)}$. Clearly, Y is complemented in X, hence X is not isomorphic to $C(\Delta)$. On the other hand, X is not isomorphic to Y since $Y^* \cong l_1 \neq X^*$. ■

Before we prove Theorem I and Theorem II in section 3 and 4 we consider in the next section special Schauder bases in arbitrary separable L_1-predual spaces and prove some technical lemmas.

2. Admissible Schauder bases in L_1-predual spaces

A separable L_1-predual X can be represented as follows: $X = \overline{\bigcup_{n \in \mathbf{N}} E_n}$, $l^n_\infty \cong E_n \subset E_{n+1}$ for all n. Let $\{e_{i,n} \mid i = 1, \cdots, n\}$ be the canonical unit vector basis of $E_n \cong l^n_\infty$. Then the unit vector basis $\{e_{i,n+1} \mid i = 1, \cdots, n+1\}$ can be taken in such a succession that

$$(*) \qquad e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}, \qquad i = 1, \cdots, n,$$

where $a_{i,n}$ are some real numbers with $\sum_{i=1}^n | a_{i,n} | \leqq 1$. Let us define $e_n = e_{n,n}$ for all $n \in \mathbf{N}$. It is easy to see that $(e_n)_{n \in \mathbf{N}}$ is a monotone basis of X, i.e. the basis constant of $(e_n)_{n \in \mathbf{N}}$ is one. We call such bases $(e_n)_{n \in \mathbf{N}}$ *admissible Schauder bases.* Define linear functionals Φ_j on X as follows:

$$\Phi_j(e_{i,n}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{for all } n \geqq j.$$

Φ_j is indeed well defined and we obtain that $\{\pm \Phi_j \mid j \in \mathbf{N}\}$ is w*-dense in the extreme point set of the dual unit ball of X, $\operatorname{ex} B(X^*)$. We call these functionals *associated* with $(e_n)_{n \in \mathbf{N}}$ ([3], [7], [8], [10]).

LEMMA 1. *Let $(e_n)_{n\in\mathbf{N}}$ be an admissible basis of an L_1-predual whose associated functionals are $(\Phi_n)_{n\in\mathbf{N}}$. Let $N\subset\mathbf{N}$. Then $Y_N=$ closed span$\{e_n \mid n\in N\}$ is an L_1-predual and $(e_n)_{n\in N}$ is an admissible basis of Y_N. Furthermore, $\Phi_{n|Y_N}$, $n\in N$, are the functionals associated with $(e_n)_{n\in N}$.*

PROOF. Let $n_1<n_2<\cdots<n_m$ be the first elements of N. We show, $e_{n_1},e_{n_2},\cdots,e_{n_m}$ span l_∞^m and e_{n_m} is the last element of the unit vector basis of l_∞^m. This is clear if $m=1$. Assume that it is true for some $m-1$ and consider $e_{n_1},e_{n_2},\cdots,e_{n_{m-1}},e_{n_m}$. Put $F_{m-1}=\operatorname{span}\{e_{n_1},e_{n_2},\cdots,e_{n_{m-1}}\}\cong l_\infty^{m-1}$. Let $\{f_1,\cdots,f_{m-1}\}$ be the unit vector basis of F_{m-1} (with $f_{m-1}=e_{n_{m-1}}$) such that

$$\Phi_{n_i}(f_j)=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases},\qquad i,j=1,\cdots,m-1.$$

Consider $E_{n_m}=\operatorname{span}\{e_1,e_2,\cdots,e_{n_m-1},e_{n_m}\}$, whose unit vector basis is $\{e_{i,n_m} \mid i\leqq n_m\}$ (i.e. $e_{n_m,n_m}=e_{n_m}$). Define $\Lambda=\{n_1,n_2,\cdots,n_{m-1}\}$. Then we have

$$f_i=e_{n_i,n_m}+\sum_{\substack{j=1\\ j\notin\Lambda}}^{n_m}\Phi_j(f_i)e_{j,n_m},\qquad i=1,\cdots,m-1.$$

Let

$$h_i=e_{n_i,n_m}+\sum_{\substack{j=1\\ j\notin\Lambda}}^{n_m-1}\Phi_j(f_i)e_{j,n_m},\qquad i=1,\cdots,m-1\quad\text{and}\quad h_m=e_{n_m}.$$

Then the h_i, $i=1,\cdots,m$, are the elements of the unit vector basis of $\operatorname{span}(F_{m-1}\cup\{e_{n_m}\})\cong l_\infty^m$ and we have

$$\Phi_{n_j}(h_i)=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases},\qquad i,j=1,\cdots,m.$$ ■

COROLLARY 2. *Let $(e_n)_{n\in\mathbf{N}}$ be an admissible basis of an L_1-predual X whose associated functionals are $(\Phi_n)_{n\in\mathbf{N}}$. Let $N\subset\mathbf{N}$ be a subset and assume that there is $0<\lambda<1$ with w*-closure$\{\Phi_n \mid n\in N\}\subset\{\pm\Phi_n \mid n\in N\}\cup\lambda B(X^*)$. Then $\operatorname{ex}B(Y_N^*)=\{\pm\Phi_{n|Y_N} \mid n\in N\}$ and hence $Y_N^*\cong l_1$.*

PROOF. $\{\pm\Phi_{n|Y_N} \mid n\in N\}$ must be w*-dense in $\operatorname{ex}B(Y_N^*)$, so we have

$$\begin{aligned}\operatorname{ex}B(Y_N^*)&\subset \text{w*-closure}\{\pm\Phi_{n|Y_N} \mid n\in N\}\cap\{y^*\in Y_N \mid \|y^*\|=1\}\\ &\subset\{\pm\Phi_{n|Y_N} \mid n\in N\}.\end{aligned}$$

Since Y_N is an L_1-predual, $Y_N^*\cong l_1$. ■

3. Proof of Theorem I

Again, let S be the Poulsen simplex. Let $(e_n)_{n\in\mathbf{N}}$ be a given admissible basis of $A(S)$. We can assume w.l.o.g. that the associated functionals $(\Phi_n)_{n\in\mathbf{N}}$ are positive in the natural order of $A(S)^* \cong L_1$, because, if $\Phi_j \not\geqq 0$ for some j, we may take $-\Phi_j$ and $-e_j$ instead of Φ_j and e_j. An easy way to construct admissible bases $(e_n)_{n\in\mathbf{N}}$ in $A(S)$ *is to consider a triangular matrix* $A = (a_{i,n})$ whose columns are dense in $\{(\alpha_i)\in l_1 \mid 0\leqq \alpha_i,\ i=1,2,\cdots,\ \Sigma_{i=1}^{\infty}\alpha_i = 1\}$ with respect to the l_1-norm. These columns define by (*) (section 2) isometric embeddings $E_n \cong l_\infty^n \to E_{n+1} \cong l_\infty^{n+1}$. The corresponding basis $(e_n)_{n\in\mathbf{N}}$, defined as in the preceding section, then satisfies $e_n \geqq 0$, $n \in \mathbf{N}$. Furthermore, it is easy to show that in this case $e_1 = 1_S$.

Let us now assume that $(e_n)_{n\in\mathbf{N}}$ is an arbitrary given admissible basis of $A(S)$ such that $e_1 = 1_S$ and the corresponding associated functionals all are positive. This is equivalent to $\Phi_j(e_1) = 1$ for all $j \in \mathbf{N}$.

We obtain closed $\operatorname{span}\{e_n \mid n > 1\} = \{f \in A(S) \mid f(s_1) = 0\}$ for some fixed $s_1 \in \operatorname{ex} S$. The positivity of the Φ_n ensures that $\hat{S} = \mathrm{w}^*\text{-closure}\{\Phi_n \mid n \in \mathbf{N}\}$ is, by evaluation, affinely homeomorphic to S. We have

$$\hat{S} = \{x^* \in A(S)^* \mid \|x^*\| = x^*(e_1) = 1\}.$$

Now, let $Q : A(S) \to A(S)$ be a bounded linear operator.

LEMMA 4. *Let $\tau > 0$. Then there is a subsequence $N \subset \mathbf{N}$ such that for $\tilde{Q} = Q$ or for $\tilde{Q} = \mathrm{id}\text{-}Q$ there exists $1/2 \leqq \kappa \leqq \|\tilde{Q}\|$ with*

(i) $\Phi_n(\tilde{Q}e_n) \in [\kappa, \kappa + \tau]$ *for all* $n \in N$,

(ii) $|\Phi_m(\tilde{Q}e_n)| \leqq 2^{-k}$ *if $n, m \in N$, $n > m$ and n is the k-th element in N,*

(iii) w*-*closure* $\{\Phi_n \mid n \in N\}$ *contains an interior point with respect to the restriction of the* w*-*topology on* $\hat{S}$.

PROOF. At first, we define a subsequence $N_1 \subset \mathbf{N}$: Since $\Phi_n(e_n) = 1$ for all $n \in \mathbf{N}$ we have $\Phi_n(Qe_n) \geqq 1/2$ or $\Phi_n((\mathrm{id}\text{-}Q)e_n) \geqq 1/2$. Let $\omega = \max(\|Q\|, \|\mathrm{id}\text{-}Q\|)$ and let I_j, $j = 1, \cdots, n$, for some $n \in \mathbf{N}$, be intervals, each of whose length is smaller than τ, such that $\bigcup_{j=1}^{n} I_j = [1/2, \omega]$. Let $W_j = \mathrm{w}^*\text{-closure}\{\Phi_n \mid \Phi_n(Qe_n) \in I_j\}$ and $W_{n+j} = \mathrm{w}^*\text{-closure}\{\Phi_n \mid \Phi_n((\mathrm{id}\text{-}Q)e_n) \in I_j\}$. Hence $\hat{S} = \bigcup_{j=1}^{2n} W_j$, since $\{\Phi_n \mid n \in \mathbf{N}\}$ is w*-dense in $\hat{S}$. From Baire's theorem it follows that there is $j_0 \in \{1, 2, \cdots, 2n\}$ such that interior $W_{j_0} \neq \emptyset$. Assume w.l.o.g. that $j_0 \leqq n$, i.e. put $\tilde{Q} = Q$. Define $N_1 = \{n \in \mathbf{N} \mid \Phi_n \in W_{j_0}\}$. N_1 satisfies (i) and (iii). Next, we choose a subsequence $N \subset N_1$ as follows: Let $U \subset W_{j_0}$ be w*-open and connected, and let $\Psi_n \in U$, $n \in \mathbf{N}$, be w*-dense in U such that $\|\Psi_{n|F}\| < 1$ for all $n \in \mathbf{N}$ with $F = \operatorname{span}\{e_n \mid n > 1\}$. Start with $1 < n_1 \in N_1$ arbitrarily. Assume that we have defined already $n_1 < n_2 < \cdots < n_k \in N_1$. There is a subsequence $M \subset N_1$, containing $n_1, n_2, \cdots, n_k$ but $1 \notin M$ such that $\Phi_m \to \Psi_k$ if $m \in M$ and $m \to \infty$ with respect to the w*-topology. Hence $Y_M^* \cong l_1$ with $Y_M = \text{closed span}\{e_m \mid m \in M\}$ (Lemma 1 and Corollary 2). There are $\mu_{i,j} \in \mathbf{R}$ such that $\Phi_{n_i} \circ Q_{|Y_M} = \Sigma_{j \in M} \mu_{i,j} \Phi_{j|Y_M}$, $i = 1, \cdots, k$, and $\Sigma_{j \in M} |\mu_{i,j}| < \infty$ for all i. Let $n_{k+1} \in M$ be such that $n_{k+1} > n_k$,

$$|\Phi_{n_i}(Qe_{n_{k+1}})| = \left| \sum_{j \in M} \mu_{i,j} \Phi_j(e_{n_{k+1}}) \right| \leqq \sum_{\substack{j \in M \\ j \geqq n_{k+1}}} |\mu_{i,j}| \leqq 2^{-k-1}$$

for all $i = 1, \cdots, k$ and

$$(+) \qquad \|(\Phi_{n_{k+1}} - \Psi_k)_{|E_k}\| \leqq k^{-1}, \qquad \text{where } E_k = \text{span}\{e_1, \cdots, e_k\}.$$

$(+)$ ensures that $U \subset \text{w*-closure}\{\Phi_{n_k} \mid k \in \mathbf{N}\}$.

Put $N = \{n_1, n_2, \cdots\}$. ∎

For the rest of this section, let us assume $\tilde{Q} = Q$.

Let $N = \{n_1, n_2, \cdots\}$ be the sequence of Lemma 4. Fix $k \in \mathbf{N}$ and consider $x = \Sigma_{j>k} t_j e_{n_j}$. Then we have, for $i < k$, $|\Phi_{n_i}(Qx)| \leqq 2^{-k} \max_{j>k} |t_j| \leqq 2^{-k+1} \|x\|$ since $\sup_n |t_n| \leqq 2 \|\Sigma_{n=1}^{\infty} t_n e_n\|$ ((e_n) is a monotone basis).

Consider the weight operator W defined by

$$W\left(\sum_{n \in \mathbf{N}} t_n e_n\right) = \sum_{n \in \mathbf{N}} t_n \Phi_n(Qe_n) e_n, \qquad t_n \in \mathbf{R}.$$

In the next lemma we show that W is well defined and bounded on a suitable subspace of $A(S)$.

Let us always retain the above specialized admissible basis $(e_n)_{n \in \mathbf{N}}$ of $A(S)$. Again define $E_n = \text{span}\{e_1, e_2, \cdots, e_n\}$ for all $n \in \mathbf{N}$. Consider $M \subseteq \mathbf{N}$. Then we call Y_M a *facial subspace* of $A(S)$ (with respect to $(e_n)_{n \in \mathbf{N}}$) if $K = \text{w*-closed conv}\{\Phi_m \mid m \in M\}$ is a face of $\hat{S}$ which is affinely homeomorphic to S, the restriction map $K \to K|_{Y_M}$ is a homeomorphism and we have $\|y\| = \sup_{k \in K} |k(y)|$ for all $y \in Y_M$. It is clear that in this case $Y_M \cong A(S)$ and $R: A(S) \to Y_M$ with $k(Rf) = k(f)$, for all $k \in K$, $f \in A(S)$, is a contractive projection.

LEMMA 5. *Let $N \subset \mathbf{N}$ be a subsequence such that the interior of* w*-*closure* $\{\Phi_n \mid n \in N\}$ *is non empty (in $\hat{S}$). Then there is a subsequence $M \subset N$ such that Y_M is a facial subspace. Moreover, if N is the sequence of Lemma 4, then for every $\varepsilon > 0$ we can arrange M such that in addition*

$$|\Phi_m(Wy) - \Phi_m(Qy)| \leqq \varepsilon \|y\|$$

for all $y \in Y_M$ and all $m \in M$.

PROOF. It suffices to assume that N is as in Lemma 4. Since the interior of w*-closure $\{\Phi_n \mid n \in N\}$ is non-empty we may assume that Φ_n, for some $n \in N$, is an interior point. Moreover, we can suppose that there is $1 > \delta > 0$ such that $x^* \in$ w*-closure $\{\Phi_k \mid k \in N\}$ whenever $x^* \in \hat{S}$ and $\|(x^* - \Phi_n)_{|E_n}\| < \delta$. Choose $M = \{m_1, m_2, \cdots\}$ by induction: Put $m_1 = n$. Assume that we have defined already $m_1 < m_2 < \cdots < m_k$ such that $\|(\Phi_{m_i} - \Phi_n)_{|E_n}\| < \delta$, $i = 1, \cdots, k$ and

$$|\Phi_{m_i}(Wy) - \Phi_{m_i}(Qy)| \leqq (1 - 2^{-k})\varepsilon \|y\|,$$

for all $i = 1, \cdots, k$ and $y \in \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_k}\}$.

(0) Let $H = \operatorname{conv}\{\Phi_{m_i} \mid i = 1, \cdots, k\}$ and choose $x_1^*, \cdots, x_r^* \in H$ such that, whenever $x^* \in H$, then there is $i \in \{1, \cdots, r\}$ with

$$\|x^* - x_i^*\| \leqq 2^{-k-1}. \tag{1}$$

At first we consider x_1^*. In view of our assumption on the Φ_{m_i} we clearly have

$$|x_1^*(Wy) - x_1^*(Qy)| \leqq (1 - 2^{-k})\varepsilon \|y\|, \tag{2}$$

for all $y \in \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_k}\}$, and $\|(x_1^* - \Phi_n)_{|E_n}\| < \delta$. There is $m_{k+1} \in N$, $m_{k+1} > m_k$, such that

$$\|(\Phi_{m_{k+1}} - x_1^*)_{|E_{m_k} + QE_{m_k}}\| \leqq \delta_1 < 2^{-k-1} \tag{3$_1$}$$

for some $0 < \delta_1$. If δ_1 is sufficiently small we have

$$\|(\Phi_{m_{k+1}} - \Phi_n)_{|E_n}\| < \delta.$$

Let $y_0 \in \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_k}\}$ and put $y = y_0 + te_{m_{k+1}}$, $t \in \mathbf{R}$. Then with (2) and (3) we have, if δ_1 was suitably small,

$$\begin{aligned} |\Phi_{m_{k+1}}(Wy) - \Phi_{m_{k+1}}(Qy)| &= |\Phi_{m_{k+1}}(Wy_0) - \Phi_{m_{k+1}}(Qy_0)| \\ &\leqq (1 - 2^{-k-1})\varepsilon \|y_0\| \\ &\leqq (1 - 2^{-k-1})\varepsilon \|y\|. \end{aligned}$$

From one of the properties of the elements in N it follows that $\Phi_{m_i}(Qe_{m_{k+1}})$ is small for all $i = 1, \cdots, k$, if m_{k+1} is large (Lemma 4(ii)). Hence if m_{k+1} is appropriately chosen we obtain:

$$\begin{aligned} |\Phi_{m_i}(Wy) - \Phi_{m_i}(Qy)| &\leqq |\Phi_{m_i}(Wy_0) - \Phi_{m_i}(Qy_0)| + |t\Phi_{m_i}(Qe_{m_{k+1}})| \\ &\leqq (1 - 2^{-k-1})\varepsilon \|y\| \end{aligned}$$

$$\text{for all } y \in \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_{k+1}}\} \quad \text{and} \quad i = 1, \cdots, k.$$

In the analogous manner treat x_2^*, i.e. choose $N \ni m_{k+2} > m_{k+1}$ large enough such that

$$(3)_2 \qquad \|(\Phi_{m_{k+2}} - x_2^*)_{|E_{m_{k+1}} + QE_{m_{k+1}}}\| \leqq \delta_2 < 2^{-k-1}$$

for suitably small $\delta_2 > 0$. This yields

$$\|(\Phi_{m_{k+2}} - \Phi_n)_{|E_n}\| < \delta \qquad \text{and}$$

$$|\Phi_{m_i}(Wy) - \Phi_{m_i}(Qy)| \leqq (1 - 2^{-k-2})\varepsilon \, \|y\|$$

$$\text{for all } y \in \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_{k+2}}\} \quad \text{and} \quad i = 1, \cdots, k+2.$$

Continue with $x_3^*, \cdots, x_r^*$. We obtain $\Phi_{m_{k+1}}, \cdots, \Phi_{m_{k+r}}$ such that for every $x^* \in H$ there is $i \in \{1, \cdots, r\}$ with

$$(4) \qquad \|(\Phi_{m_{k+i}} - x^*)_{|E_{m_k}}\| \leqq 2^{-k}$$

and for every $j = 1, 2, \cdots, k+r$ there is $x^* \in H$ with

$$(5) \qquad \|(\Phi_{m_j} - x^*)_{|E_{m_k}}\| \leqq 2^{-k}.$$

Furthermore,

$$|\Phi_{m_i}(Wy) - \Phi_{m_i}(QY)| \leqq (1 - 2^{-k-r})\varepsilon \, \|y\|$$

for all $y \in \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_{k+r}}\}$ and $i = 1, \cdots, k+r$.

Then go back to (0) and do the next step in our induction.

Define $M = \{m_1, m_2, \cdots\}$. Let K = w*-closure $\operatorname{conv}(\{\Phi_j \mid j \in M\})$. Our construction and (4) ensure that, whenever $m \in \mathbf{N}$ and $x^* \in K$, then there is some $j \in M$ with $\|(\Phi_j - x^*)_{|E_m}\| \leqq 2^{-m}$, hence ex K is w*-dense in K. Furthermore, by (5), for each $x^* \in K$ and $m_k \in M$ there is $y^* \in \operatorname{conv}(\{\Phi_{m_1}, \cdots, \Phi_{m_k}\})$ with $\|(x^* - y^*)_{|E_{m_k}}\| \leqq 2^{-k-1}$. This means, K is a face: Indeed, let $x^* \in K$ and assume $x^* = 1/2\, u^* + 1/2\, v^*$, $\|u^*\|, \|v^*\| \leqq 1$. Let $y_k^* \in \operatorname{conv}\{\Phi_{m_1}, \cdots, \Phi_{m_k}\})$, $k \in \mathbf{N}$, such that $y_k^* \to^{w^*} x^*$. Since $\operatorname{conv}(\{\Phi_{m_1|E_{m_k}}, \cdots \Phi_{m_k|E_{m_k}}\})$ is a face of $B(E_{m_k}^*) = B(l_1)$, there are u_k^*, $v_k^* \in \operatorname{conv}(\{\Phi_{m_1}, \cdots, \Phi_{m_k}\})$ such that $u_k^* \xrightarrow{w^*} u^*$, $v_k^* \xrightarrow{w^*} v^*$. Hence $u^*, v^* \in K$.

Put $D_k = \operatorname{span}\{e_{m_1}, e_{m_2}, \cdots, e_{m_k}\}$. If $x^*, y^* \in K$, $x^* \neq y^*$, then there is $m \in \mathbf{N}$ and $x \in E_m$, $\|x\| \leqq 1$, such that $x^*(x) \neq y^*(x)$. Let k be so large that $m \leqq m_k$ and $2^{-k+3} < |x^*(x) - y^*(x)|$. Let $\{f_1, \cdots, f_k\}$ be the unit vector basis of $D_k \cong l_\infty^k$ and define $x_k = \sum_{i=1}^k \Phi_{m_i}(x) f_i$. Since there are $x_0^*, y_0^* \in \operatorname{conv}(\{\Phi_{m_1}, \Phi_{m_2}, \cdots, \Phi_{m_k}\})$ with $\|(x_0^* - x^*)_{|E_{m_k}}\|$, $\|(y_0^* - y^*)_{|E_{m_k}}\| \leqq 2^{-k+1}$ we obtain

$$|x^*(x) - x^*(x_k)| \leqq |x_0^*(x) - x_0^*(x_k)| + 2^{-k+2} = 2^{-k+2}$$

and

$$|y^*(x)-y^*(x_k)|\leqq|y_0^*(x)-y_0^*(x_k)|+2^{-k+2}=2^{-k+2}.$$

Hence $|x^*(x_k)-y^*(x_k)|\geqq|x^*(x)-y^*(x)|-2^{-k+3}>0$, i.e. $x^*_{|Y_M}\neq y^*_{|Y_M}$. This proves Lemma 5. ∎

Let $\delta>0$, $n_0\in\mathbf{N}$ and assume that $E\subset A(S)$ is a finite dimensional subspace. Let $N=\{n\in\mathbf{N}\mid\|(\Phi_n-\Phi_{n_0})_{|E}\|<\delta\}$. Then in view of the density of ex $\hat{S}$ in $\hat{S}$ we have that w*-closure $\{\Phi_n\mid n\in N\}$ contains an interior point. Therefore Lemma 5 is applicable to such a sequence N.

Let N be again as in Lemma 4 and let $M\subset N$ be the corresponding subsequence of Lemma 5. W maps Y_M on a dense subspace of Y_M. If W is an isomorphism then it is surjective and in this case Theorem 1 follows immediately from Lemma 5 (with $Y=Y_M$). Unfortunately, W need not be an isomorphism. So the difficulty of the following proof is to construct a new admissible basis. Then we have to use the arguments of the proof of Lemma 5 again.

Now, we prove Theorem I. Let $0<\delta<1$ be arbitrary. Consider the subsequence N of Lemma 4 with some $\tau>0$ depending on δ (to be determined later).

Applying Lemma 5 we may restrict our considerations to a suitable facial subspace Y_M of Y_N, with all the properties mentioned in Lemma 5. Here we choose $0<\varepsilon<1/2(2+\delta)$. For simplicity we may assume w.l.o.g. that $M=\mathbf{N}$ (i.e. $Y_M\cong A(S)$). We have then $\Phi_n(Qe_n)\in[\kappa,\kappa+\tau]$ for all $n\in\mathbf{N}$, where $\kappa\geqq\frac{1}{2}$ is the constant of Lemma 4. Furthermore we have

$$|\Phi_n(Wy)-\Phi_n(Qy)|\leqq\varepsilon\|y\|\qquad\text{for all } n\in\mathbf{N},\quad y\in A(S)$$

and

$$|\Phi_m(Qx)|\leqq 2^{-n+1}\|x\|\qquad\text{for all } x\in\operatorname{span}\{e_{n+1},e_{n+2},\cdots\}\quad\text{and}\quad m<n.$$

We consider two alternatives: At first, assume that there is facial subspace Y whose facial projection is R, such that

$$\|RQy\|\geqq(2+\delta)^{-1}\qquad\text{whenever } y\in Y\quad\text{and}\quad\|y\|=1.$$

Here, from our assumptions, we infer $\|(W-RQ)_{|Y}\|\leqq\varepsilon$. $T=RQ_{|Y}$ is invertible and therefore W is invertible. Hence both operators are surjective. We have $\|T^{-1}\|\leqq 2+\delta$. Define $R_1:A(S)\to QY$ by $QT^{-1}R$. If δ is sufficiently small, we are done.

Assume now, that for every facial subspace Y there is $y\in Y$ with $\|y\|=1$ and $\|RQy\|<(2+\delta)^{-1}$. We construct then a subspace $Y\cong A(S)$ and a contractive projection $R:A(S)\to Y$ such that $\|\kappa\,\mathrm{id}_Y-RQ_{|Y}\|\leqq\delta$. Then we obtain

$1/2\|y\| \leqq \kappa\|y\| \leqq \|RQy\| + \delta\|y\|$ for all $y \in Y$. Hence again, $T = RQ_{|Y}$ is invertible if δ is small enough and we can apply the same argument as before. We define Y in a manner which is similar to that used in the proof of Lemma 5: Let $\alpha_{i,k}$, $i = 0, 1, 2, \cdots, k$, $k \in \mathbf{N}$, be non-negative real numbers such that $\Sigma_{i=0}^{k} \alpha_{i,k} = 1$ for all $k \in \mathbf{N}$, the l_1-closure of $\{(\alpha_{1,k}, \alpha_{2,k}, \cdots, \alpha_{k,k}, 0, 0, \cdots) \mid k \in \mathbf{N}\}$ contains $\{(\alpha_i) \in l_1 \mid \alpha_i \geqq 0,\ i = 1, 2, \cdots, \Sigma_{i=1}^{\infty} \alpha_i = 1\}$ and

(0) $$2^{-k-1} \leqq \alpha_{0,k} \leqq 2^{-k} \qquad \text{for all } k \in \mathbf{N}.$$

Put $F_0 = \{0\}$. Assume that we have defined already $F_k = \operatorname{span}\{f_1, \cdots, f_k\} \cong l_\infty^k$ where $\{f_1, \cdots, f_k\}$ is an admissible basis of l_∞^k with

(1) $$\{f_1, \cdots, f_k\} \subset \operatorname{span}\{e_2, e_3, \cdots, e_n\} \qquad \text{for some } n \in \mathbf{N}.$$

Assume furthermore that we have defined already $m_1 < m_2 < \cdots < m_k$ such that $\Phi_{m_1}, \Phi_{m_2}, \cdots, \Phi_{m_k}$ are associated with $f_1, f_2, \cdots, f_k$. Moreover suppose that $R_k : E_n \to F_k$ is a contractive projection such that $\Phi_{m_j}(x) = \Phi_{m_j}(R_k x)$ for all $j = 1, \cdots, k$ and $x \in E_n$. Finally assume

(2) $$|\kappa \Phi_{m_i}(f) - \Phi_{m_i}(Qf)| \leqq \left(\sum_{j=1}^{k} 2^{-j}\right) \delta \|f\| \qquad \text{for all } f \in F_k, \quad i = 1, \cdots, k.$$

Let $0 < \rho < \min(2^{-k-4}\|Q\|^{-1}\delta, 2^{-k-4}\delta, \tau)$. Let $p \in \mathbf{N}$ be such that $p > n$,

(3) $$|\Phi_{m_i}(Qx)| \leqq \rho\|x\| \qquad \text{for all } i = 1, \cdots, k \quad \text{and} \quad x \in \operatorname{span}\{e_q \mid q \geqq p\}$$

and

(4) $$\left\| \left(\Phi_p - \left(\alpha_{0,k}\Phi_1 + \sum_{i=1}^{k} \alpha_{i,k}\Phi_{m_i} \right) \right)_{|E_n + QE_n} \right\| \leqq \rho.$$

By Lemma 5 there is a subsequence $N \subset \{q \in \mathbf{N} \mid q \geqq p\}$ such that Y_N is a facial subspace and

(5) $$\|(\Phi_q - \Phi_p)_{|E_p + QE_p}\| \leqq \rho^2 \qquad \text{for all } q \in N.$$

By our assumption, there is $y \in Y_N$ such that $\|y\| = 1$ but $\|RQy\| < (2+\delta)^{-1}$. We can assume without loss of generality that $y \in \operatorname{span}\{e_q \mid q \in N\}$, i.e. there is $q \in N$ with $|\Phi_q(y)| = 1$ and $|\Phi_q(Qy)| < (2+\delta)^{-1}$. Suppose $\Phi_q(y) = 1$ (otherwise take $-y$). Say $y \in G = \operatorname{span}(E_p \cup \{e_j \mid j \in N, j \leqq r\})$ for some $r > q$. In view of Lemma 1, G is an l_∞^m-space where $m = \dim G$. Put $f = \lambda e_p + (1-\lambda)y$ where $0 \leqq \lambda \leqq 1$ is so that

(6) $$\kappa \geqq \Phi_q(Qf) \geqq \kappa - \rho.$$

By (5) we obtain $\kappa + 2\tau \geqq \kappa + \tau + \rho \geqq \Phi_p(Qe_p) + \rho^2 \| Q \| \geqq \Phi_q(Qe_p) \geqq \Phi_p(Qe_p) - \rho^2 \| Q \| \geqq \kappa - \rho \geqq \kappa - \tau \geqq 1/2 - \tau$. If τ is sufficiently small (such that, for instance, $(1/2 - \tau) - (2 + \delta)^{-1} > 12\tau$), we have $\lambda \geqq 3/4 \geqq (2(1 - \rho^2))^{-1}$. Hence $\Phi_j(f) \geqq 0$ for all $j \leqq r$, $j \in \mathbf{N}$ ((5)). Moreover, $\Phi_i(f) = 0$, $i = 1, \cdots, p - 1$, $1 \geqq \Phi_p(f) \geqq 0$, $1 \geqq \Phi_q(f) \geqq 1 - \rho^2$ and $|\Phi_{m_i}(Qf)| \leqq \rho$ for all $i = 1, \cdots, k$ (in view of (3)).

Define $m_{k+1} = q$ and $f_{k+1} \in G$ by $\Phi_q(f_{k+1}) = 1$ and $\Phi_j(f_{k+1}) = (1 - \rho)\Phi_j(f)$, if $j = 1, \cdots, p$, or $j \in N$, $j \leqq r\, j \neq q$. Let $f_{i,k}$, $i = 1, \cdots, k$, be the elements of the unit vector basis of F_k. Then we have for $j \in N$, $j \leqq r$, $j \neq q$ and for $j = p$:

$$\sum_{i=1}^{k} |\Phi_j(f_{i,k}) - \Phi_q(f_{i,k})\Phi_j(f_{k+1})| + |\Phi_j(f_{k+1})|$$

$$\leqq \sum_{i=1}^{k} |\Phi_j(f_{i,k}) - \Phi_q(f_{i,k})| + (1 - \Phi_j(f_{k+1})) \sum_{i=1}^{k} |\Phi_q(f_{i,k})| + |\Phi_j(f_{k+1})|$$

$$\leqq 2\rho^2 + \sum_{i=1}^{k} |\Phi_q(f_{i,k})| + \left(1 - \sum_{i=1}^{k} |\Phi_q(f_{i,k})|\right)(1 - \rho)$$

$$= 2\rho^2 + 1 - \rho\left(1 - \sum_{i=1}^{k} |\Phi_q(f_{i,k})|\right) \leqq 2\rho^2 + 1 - \rho(2^{-k-1} - \rho - \rho^2) \leqq 1$$

in view of (0), (4), (5) and the choice of ρ. (Recall that $\Phi_j(f_{k+1}) \geqq 0$ for all $j = 1, \cdots, p$ and for $j \in N$, $j \leqq r$. Furthermore, in view of (1) we have $\Phi_1(f_{i,k}) = 0$ for all $i = 1, \cdots, k$. Hence, by (4), (5) and (0), $1 - \Sigma_{i=1}^{k} |\Phi_q(f_{i,k})| \geqq 2^{-k-1} - \rho - \rho^2$.) Thus $f_{i,k} - \Phi_q(f_{i,k})f_{k+1}$, $i = 1, \cdots, k$, and f_{k+1} are the elements of the unit vector basis of $F_{k+1} = \operatorname{span}\{f_1, f_2, \cdots, f_{k+1}\} \cong l_\infty^{k+1}$, since $F_{k+1} \subset G$, i.e.

$$\| x \| = \sup\{|\Phi_j(x)| \mid j = 1, \cdots, p, \text{ or } j \in N, j \leqq r\} \qquad \text{for all } x \in F_{k+1}.$$

By (6) we obtain

$$(7) \qquad |\Phi_q(Qf_{k+1}) - \kappa| \leqq (1 + \| Q \|)\rho \leqq 2^{-k-3}\delta \qquad \text{since } \| f - f_{k+1} \| \leqq \rho.$$

The latter inequality also yields ((3)):

$$(8) \qquad |\Phi_{m_i}(Qf_{k+1})| \leqq (1 + \| Q \|)\rho \leqq 2^{-k-3}\delta \qquad \text{for all } i = 1, \cdots, k.$$

Hence we have ($q = m_{k+1}$)

$$(9) \qquad |\kappa\Phi_{m_i}(y) - \Phi_{m_i}(Qy)| \leqq \left(\sum_{j=1}^{k+1} 2^{-j}\right) \delta \| y \|, \qquad i = 1, \cdots, k + 1,$$

if $y = \Sigma_{j=1}^{k+1} t_j f_j$. This follows for $i = 1, \cdots, k$ from (2), (8), and for $i = k + 1$ from (2), (4), (5) and (7). (Note that $\| \Sigma_{j=1}^{k} t_j f_j \| \leqq \| y \|$ since $\{f_1, \cdots, f_{k+1}\}$ is a monotone basis.) Let $x \in E_r$ and define $R_{k+1}x = \sum_{i=1}^{k} \Phi_{m_i}(x)(f_{i,k} - \Phi_q(f_{i,q})f_{k+1}) + \Phi_{m_{k+1}}(x)f_{k+1}$. We obtain $\Phi_{m_i}(x) = \Phi_{m_i}(R_{k+1}x)$ for all $i = 1, \cdots, k+1$. Furthermore, if $y \in E_n$, then we have

$$
\begin{aligned}
\| R_k x - R_{k+1} x \| &= \left\| \sum_{i=1}^{k} \Phi_{m_i}(x) f_{i,k} - \sum_{i=1}^{k} \Phi_{m_i}(x)(f_{i,k} - \Phi_q(f_{i,k}) f_{k+1}) - \Phi_{m_{k+1}}(x) f_{k+1} \right\| \\
&\leqq 2^{-k+1} \| x \|
\end{aligned} \tag{10}
$$

since, with $q = m_{k+1}$,

$$
\begin{aligned}
&\left\| \left(\Phi_q - \sum_{i=1}^{k} \Phi_q(f_{i,k}) \Phi_{m_i} \right)_{|E_n} \right\| \\
&\quad \leqq \left\| \Phi_q - \left(\alpha_{0,k} \Phi_1 + \sum_{i=1}^{k} \alpha_{i,k} \Phi_{m_i} \right) \right\| + \alpha_{0,k} + \left\| \sum_{i=1}^{k} (\alpha_{i,k} - \Phi_q(f_{i,k})) \Phi_{m_i} \right\| \\
&\quad \leqq \rho^2 + \rho + \sum_{i=1}^{k} | \Phi_q(f_{i,k}) - \alpha_{i,k} | + \alpha_{0,k} \leqq 2\rho^2 + 2\rho + 2^{-k} \leqq 2^{-k+1}
\end{aligned}
$$

(cf. (4), (5) and (0)). Define $Y = \text{closed span}\{f_1, f_2, \cdots\}$. $R : A(S) \to Y$ with $Rx = \lim_{k \to \infty} R_k x$ for all $x \in \bigcup_{n \in \mathbf{N}} E_n$ is a contractive projection with $\Phi_{m_i}(Rx) = \Phi_{m_i}(x)$ for all i (cf. (10)). Then (9) yields $\| \kappa \, \mathrm{id}_Y - RQ_{|Y} \| \leqq \delta$. The choice of the $\alpha_{i,k}$ asserts that $\sum_{i=1}^{k} \alpha_{i,k} \to 1$ if $k \to \infty$. Moreover, we have, by construction,

$$
\| \Phi_{m_{k+1}|F_k} \| \geqq 1 - 2^{-k} - \rho - \rho^2 \geqq 1 - 2^{-k+1} \qquad \text{for all } k.
$$

Hence, $\| \Phi_{m_h|F_k} \| \geqq \prod_{j=k-1}^{\infty} (1 - 2^{-j})$ for all $h \geqq k$. This means that w*-closure$\{\Phi_{m_i|Y} \mid i \in \mathbf{N}\} \subset \{y^* \in Y^* \mid \|y^*\| = 1\}$ and Y is a simplex space where the corresponding simplex is the Poulsen simplex. ■

It should be pointed out that Theorem I remains true if we replace $A(S)$ by $C(\Delta)$, Δ the Cantor set. Here we can consider the following admissible basis: Let $A = (a_{i,n})$ be a matrix whose columns are the elements of the usual unit vector basis of l_1 and assume that each of these elements appears infinitely many times among the columns of A. The corresponding numbers $a_{i,n}$ define, according to (*), section 2, isometric embeddings $l_\infty^n \cong E_n \to E_{n+1} \cong l_\infty^{n+1}$. We have $\overline{\bigcup_{n \in \mathbf{N}} E_n} = C(\Delta)$ ([7]). Take the corresponding admissible basis $(e_n)_{n \in \mathbf{N}}$. The associated functionals $(\Phi_n)_{n \in \mathbf{N}}$ here satisfy: For every $k, m \in \mathbf{N}$ there is $n \geqq m$ such that $\Phi_{n|E_m} = \Phi_{k|E_m}$. For this basis we can apply the same argument as in the proof of Theorem I with only minor modifications. In particular, in the preceding proof we may take $\rho = 0$ which simplifies the arguments considerably.

4. Proof of Theorem II

The proof of Theorem II is an easy consequence of Pelczynski's decomposition method. Let X_n, $n \in \mathbf{N}$, be Banach spaces. Then we define

$$\left(\bigoplus_{n=1}^{\infty} X_n\right)_{(c_0)} = \left\{(x_n)_{n\in\mathbf{N}} \,\middle|\, x_n \in X_n,\ \lim_{n\to\infty} \|x_n\| = 0\right\}$$

and consider the norm $\|(x_n)_{n\in\mathbf{N}}\| = \sup_{n\in\mathbf{N}} \|x_n\|$.

Lemma 6. *$A(S)$, S the Poulsen simplex, is isomorphic to $(\bigoplus_{n=1}^{\infty} A(S))_{(c_0)}$.*

Proof. Put $Z = (\bigoplus_{n=1}^{\infty} A(S))_{(c_0)}$. It is easy to show that Z is an L_1-predual space. Hence Z can be regarded as a subspace of $A(S)$ such that there is a contractive projection $P: A(S) \to Z$ and (id-P)($A(S)$) is isomorphic to $A(S)$. Hence we have

$$A(S) = (\text{id-}P)(A(S)) \oplus Z \sim A(S) \oplus \left(\bigoplus_{n=1}^{\infty} A(S)\right)_{(c_0)} \sim \left(\bigoplus_{n=1}^{\infty} A(S)\right)_{(c_0)}. \quad \blacksquare$$

Proof of Theorem II. Let $\tilde{Q} = Q$. By Theorem I there is a complemented subspace $W \subset QA(S)$ such that $W \sim A(S)$, hence we can write $QA(S) = W \oplus Z$. Furthermore, $A(S) = U \oplus QA(S)$. We have $A(S) \oplus QA(S) = A(S) \oplus W \oplus Z \sim W \oplus Z = QA(S)$ since $A(S) \oplus W \sim W$. Furthermore

$$\begin{aligned} A(S) \oplus QA(S) &\sim \left(\bigoplus_{n=1}^{\infty} A(S)\right)_{(c_0)} \oplus QA(S) \\ &= \left(\bigoplus_{n=1}^{\infty} U\right)_{(c_0)} \oplus \left(\bigoplus_{n=1}^{\infty} QA(S)\right)_{(c_0)} \oplus QA(S) \\ &\sim \left(\bigoplus_{n=1}^{\infty} U\right)_{(c_0)} \oplus \left(\bigoplus_{n=1}^{\infty} QA(S)\right)_{(c_0)} \sim A(S). \end{aligned}$$

This proves Theorem II. ■

References

1. D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math. **27** (1977), 64–92.

2. Y. Benyamini and J. Lindenstrauss, *A predual of l_1 which is not isomorphic to a $C(K)$-space*, Israel J. Math. **13** (1972), 246–259.

3. A. J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1-spaces and their representing matrices*, Acta Math. **126** (1971), 165–193.

4. J. Lindenstrauss, G. Olsen and Y. Sternfeld, *The Poulsen simplex*, Ann. Inst. Fourier (Grenoble) **28** (1978), 91–114.

5. J. Lindenstrauss and A. Pelczynski, *Contributions to the theory of the classical Banach spaces*, J. Functional Analysis **8** (1971) 225–249.

6. W. Lusky, *The Gurarij spaces are unique*, Arch. Math. **27** (1976), 627–635.

7. W. Lusky, *On separable Lindenstrauss spaces*, J. Functional Analysis **26** (1977), 103–120.

8. E. Michael and A. Pelczynski, *Separable Banach spaces which admit l_n^∞ approximation*, Israel J. Math. **4** (1966), 189–198.

9. P. Wojtaszczyk, *Some remarks on the Gurarij space*, Studia Math. **41** (1972), 207–210.

10. M. Zippin, *On some subspaces of Banach spaces whose duals are L_1-spaces*, Proc. Amer. Math. Soc. **23** (1969), 378–385.

GESAMTHOCHSCHULE PADERBORN
FACHBEREICH 17, WARBURGER STRAßE 100
4790 PADERBORN, WEST GERMANY